

Strassen's LIL for the Lorenz Curve

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We prove Strassen's law of the iterated logarithm for the Lorenz process assuming that the underlying distribution function F and its inverse F^{-1} are continuous, and the moment $\mathbf{E}X^{2+\varepsilon}$ is finite for some $\varepsilon > 0$. Previous work in this area is based on assuming the existence of the density $f := F'$ combined with further assumptions on F and f . Being based only on continuity and moment assumptions, our method of proof is different from that used previously by others, and is mainly based on a limit theorem for the (general) integrated empirical difference process. The obtained



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1. INTRODUCTION AND THE MAIN RESULT

Let X be a non-negative random variable with distribution function F . We assume throughout that the mean $\mu := \mathbf{E}X$ is finite and positive. The Lorenz curve corresponding to the random variable X , denoted by L_F , is defined (cf. Gastwirth, 1971) by the formula

$$t \mapsto L_F(t) := \frac{1}{\mu} \int_0^t F^{-1}(s) ds, \quad 0 \leq t \leq 1,$$

where F^{-1} denotes the left-continuous inverse of F .

In econometrics it is customary to interpret $L_F(t)$ as the proportion of total amount of “wealth” that is owned by the least fortunate $t \times 100$ percent of a “population.” For some details on the variety of situations where estimating the curve L_F is of importance, we may refer, for example, to:

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- (a) measurements of income and wealth—Atkinson (1970), Dasgupta, Sen, and Starrett (1973), Gastwirth (1971, 1972, 1988a, 1988b), Hart (1971, 1975), Sen (1973, 1974, 1976);
- (b) school segregation—Alker (1965);
- (c) antitrust and industrial concentration—Hart (1971, 1975);
- (d) one person one vote cases—Gastwirth (1988a, 1988b);
- (e) fisherman's luck—Thompson (1976);
- (f) bibliography—Leimkuhler (1967);
- (g) publishing productivity among scientists—Goldie (1977);

etc. Lorenz curves have been in use for more than 90 years (cf. Lorenz, 1905).

The empirical counterpart, denoted by L_n , to the Lorenz curve L_F is defined (cf. Gastwirth, 1971, 1972) as follows: Let X_1, \dots, X_n be independent copies of X , and let F_n be the empirical (right-continuous) distribution function based on X_1, \dots, X_n . If we now denote the left-continuous inverse of F_n by F_n^{-1} , then the empirical Lorenz curve L_n is the function

$$t \mapsto L_n(t) := \frac{1}{\mu_n} \int_0^t F_n^{-1}(s) ds, \quad 0 \leq t \leq 1,$$

where μ_n stands for the empirical mean of the random sample X_1, \dots, X_n .

In this paper we study Strassen's law of the iterated logarithm for the empirical Lorenz process

$$l_n := \sqrt{n} \{L_n - L_F\}.$$

Before formulating our main result, we need to introduce further notations.

Let \mathcal{H} denote the so called Finkelstein class (cf. Finkelstein, 1971) consisting of all absolutely continuous functions $h: [0, 1] \rightarrow \mathbf{R}$ such that $h(0) = 0 = h(1)$ and $\int_0^1 \{h'(s)\}^2 ds \leq 1$.

We use $D[0, 1]$ to denote the set of all left-continuous functions on $[0, 1]$ that have right-hand limits at each point.

Let \mathcal{L} be the set $\{l_h: h \in \mathcal{H}\}$, where

$$l_h(t) := -\frac{1}{\mu} \int_0^{F^{-1}(t)} h \circ F(x) dx + \frac{1}{\mu} L_F(t) \int_0^\infty h \circ F(x) dx.$$

Throughout this paper “wrt” stands for “with respect to,” and $\|\cdot\|$ denotes the sup-norm $\sup\{\cdot(t): t \in (0, 1)\}$.

The main aim of this note is to prove the following Strassen's law of the iterated logarithm for the empirical Lorenz process l_n .

MAIN THEOREM. *Assume the two conditions: (A1) F and F^{-1} are continuous, and (A2) $\mathbf{E}X^{2+\varepsilon} < \infty$ on for some $\varepsilon > 0$. Then*

$$l_n / \sqrt{2 \log \log n} \rightsquigarrow \mathcal{L} \quad \text{a.s. wrt } \|\cdot\| \quad \text{on } D[0, 1].$$

We proceed with some historical remarks concerning rates of strong consistency for the empirical Lorenz curve L_n . Assuming the finiteness of the first moment $\mu > 0$, Gail and Gastwirth (1978) obtained pointwise strong consistency of L_n . Under the same assumption, Goldie (1977) proved strong consistency of the Lorenz curve L_n uniformly over the interval $[0, 1]$, that is to say, Goldie (1977) proved the Glivenko–Cantelli type result

$$\|L_n - L_F\| \rightarrow 0, \quad n \rightarrow \infty,$$

almost surely. Together with other processes of similar vein, and assuming the existence of the density $f := F'$, as well as further assumptions on f , M. Csörgő, S. Csörgő, and Horváth [CsCsH] (1986) established a strong invariance principle for the Lorenz process l_n by which they easily concluded the following (right) rate of strong consistency (cf. Corollary 11.4 on page 96 therein) that can be considered as the first LIL type result for l_n .

THEOREM 1.1 (CsCsH, 1986). *Assume the two conditions: (B1) F is absolutely continuous and the density f is positive on the interior of the support of F ; (B2) for some $\alpha, \beta \in [0, \frac{3}{2})$*

$$\mathcal{J}\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) := \sup_{0 < t < 1} \frac{t^\alpha(1-t)^\beta}{f(F^{-1}(t))} < \infty.$$

Then, almost surely,

$$\limsup_{n \rightarrow \infty} \|l_n\| / \sqrt{2 \log \log n} \leq A_1(F)$$

with the constant

$$A_1(F) := \frac{2\sqrt{2}}{\mu} \int_0^1 \left\{ s(1-s) \log \log \frac{1}{s(1-s)} \right\}^{1/2} dF^{-1}(s).$$

It is easy to see that assumption (B2) implies (A2). Consequently, Theorem 1.1 follows from our Main Theorem.

The first “real” law of the Iterated logarithm in Strassen’s LIL form for the Lorenz process I_n was proved in two versions by Rao and Zhao (1995). We restate their two LIL results as the following two theorems.

THEOREM 1.2 (Rao and Zhao, 1995). *Under the assumptions (B1) and (B2), the statement of the Main Theorem holds true.*

THEOREM 1.3 (Rao and Zhao, 1995). *Assume the conditions: (C1) F is twice differentiable on the interior I of the support of F , and the density f is positive on I ;*

(C2) (i) *for some $\gamma > 0$,*

$$\sup_{x \in I} F(x)(1 - F(x)) |f'(x)|/f^2(x) \leq \gamma,$$

(ii) *for some $\lambda \in (0, \frac{1}{2})$*

$$\int_0^\infty (1 - F(x))^{1/2-\lambda} dx < \infty,$$

(iii) $\int_{1-\delta}^1 F^{-1}(s) dx = O(\sqrt{\delta \log \log (1/\delta)}), \delta \rightarrow 0.$

Then the statement of the Main Theorem holds true.

As was correctly indicated in Rao and Zhao (1995), these two theorems (i.e., Theorems 1.2 and 1.3) are, in general, different results in that their conditions cannot really be compared. Both of them are corollaries, however, to our Main Theorem. In the case of Theorem 1.2 this is mainly a consequence of the implication $(B2) \Rightarrow (A2)$, while in the case of Theorem 1.3 we have that $(C2, ii) \Rightarrow (A2)$. We emphasize again that our Main Theorem does not require absolute continuity of F , which is a requirement in the so far known LIL Theorems 1.1–1.3.

We conclude this section with several remarks.

Remark 1.1. A careful and tedious inspection of the original proof of the Main Theorem in M. Csörgő and Zitikis (1995b) shows that assumption $(A2)$ could possibly be replaced by the following: $(A2)'$

$$\int_0^\infty \sqrt{1 - F(x)} dx < \infty,$$

and the result of the Main Theorem would continue to hold true. Although this replacement would lead to a somewhat stronger result due to $(A2) \Rightarrow (A2)' \Rightarrow (EX^2 < \infty)$, we retained assumption $(A2)$ deliberately in the statement and the proof of the Main Theorem since it seems to us that,

just as for the approximation in probability of the process I_n by appropriate Gaussian processes (cf. Theorem 11.2 of CsCsH, 1986), the optimal assumption for the validity of the Main Theorem is $\mathbf{E}X^2 < \infty$, that is to say, assumption (A2) with $\varepsilon = 0$. Thus, changing assumption (A2) to (A2)' can be somewhat misleading. On the other hand, all our attempts to replace (A2) by $\mathbf{E}X^2 < \infty$ in the Main Theorem have failed so far.

Remark 1.2. At this stage it is not clear to us what one should do in order to relax (or get rid of) assuming continuity of the distribution function F and its inverse F^{-1} . The notions presented in Remark 3 of Major and Rejtő (1988) might turn out to be helpful in finding a solution of this problem.

Remark 1.3. From the Main Theorem we easily conclude the following result

$$\limsup_{n \rightarrow \infty} \|I_n\| / \sqrt{2 \log \log n} = A_0(F) := \sup\{\|I_h\| : h \in \mathcal{H}\} \quad \text{a.s.}$$

It is easy to check that $A_0(F) < A_1(F)$. Therefore, having proved Strassen's LIL for the Lorenz process, we may now derive a.s. better confidence bands than those obtainable from Theorem 1.1 (cf. M. Csörgő and Zitikis, 1996, for more details on the subject).

Remark 1.4. The ideas presented in, and the method of proof of, M. Csörgő and Zitikis, 1994a, can be used to prove weighted approximation results for the Lorenz process, and thus construct confidence bands for the (theoretical) Lorenz curve L_F as well (cf. M. Csörgő and Zitikis, 1996). These goals can also be achieved indirectly by using results of M. Csörgő and Zitikis (1995a) and the representation of Chandra and Singpurwalla (1978) (cf. Eq. (6) on page 776 of Shorack and Wellner, 1986, for a convenient reference)

$$L_F(F(x)) = 1 - (1 - F)(x))\{M_F(x) + x\}/\mu, \quad (1.1)$$

where $x \mapsto M_F(x) := \mathbf{E}\{X - x | X > x\}$ is the mean residual life function. On the other hand, representation (1.1) can be used to derive Strassen's LIL for mean residual life processes via utilizing now the Main Theorem here established for the Lorenz process. Furthermore, based on these observations, and on having

$$T_F(t) = \mu L_F(t) + (1 - t) F^{-1}(t), \quad (1.2)$$

another result from Chandra and Singpurwalla (1978) (cf. Eq. (7) on page 776 of Shorack and Wellner, 1986, for a convenient reference) concerning the total time on test function T_F , one can also have Strassen's LIL, weak

approximation results, confidence bands, etc., for the total time on test function T_F as well. In addition to these notions concerning T_F , we note also that, with emphasis put on Strassen's LIL, the same remarks are applicable to the Lorenz process of order ν , the Shannon process, as well as to the empirical redundancy process and some others of similar vein. For definitions and a first unified treatment of strong and weak approximations of all these processes, we refer to CsCsH (1986) and to Shorack and Wellner (1986) for further related results and discussions. In this regard we also note results by M. Csörgő and Horváth (1989), where confidence bands with prescribed confidence levels are constructed for the quantile function F^{-1} without assuming the existence of the density function f , which is not assumed in this paper either.

2. PROOF OF THE MAIN THEOREM

An elementary computation shows that

$$\begin{aligned} L_n(t) - L_F(t) &= \frac{1}{\mu_n} \int_0^t \{F_n^{-1}(s) - F^{-1}(s)\} ds - \frac{\mu_n - \mu}{\mu_n} L(t) \\ &= \frac{1}{\mu_n} \int_0^t \{F_n^{-1}(s) - F^{-1}(s)\} ds \\ &\quad - \frac{L(t)}{\mu_n} \int_0^1 \{F_n^{-1}(s) - F^{-1}(s)\} ds, \end{aligned} \quad (2.1)$$

where the last equality holds true because of

$$\mu_n - \mu = \int_0^1 \{F_n^{-1}(s) - F^{-1}(s)\} ds.$$

Having representation (2.1), one's natural inclination is to make use of the theory and assumptions of general empirical quantile processes as in M. Csörgő and Révész (1978, 1981), and this is, in fact, the very route Rao and Zhao (1995) took in proving their Theorem 1 (\equiv Theorem 1.3 in the present paper). Another inviting way is to use the strong approximation of the process ℓ_n given in Theorem 11.3 of CsCsH (1986). Indeed, Rao and Zhao (1995) based their Theorem 2 (\equiv Theorem 1.2 above) on the latter strong invariance principle. In retrospect, however, quantile methods in this context appear to have been somewhat misleading in that, roughly speaking, integrals of quantile processes are almost equal to integrals of their corresponding empirical processes. Consequently the, in general, less restrictive theory and methods of (weighted) empirical processes can be

used instead, and this, in turn, should result in stronger results than those obtainable via a direct use of quantile processes and methods. As a preliminary support of these claims, we state the following easy-to-prove equality:

$$\int_0^1 \{F_n^{-1}(s) - F^{-1}(s)\} ds = - \int_0^\infty \{F_n(x) - F(x)\} dx.$$

The problem of showing that the “remainder” term $V_n(t)$ in the “expansion”

$$\int_0^t \{F_n^{-1}(s) - F^{-1}(s)\} ds = - \int_0^{F^{-1}(t)} \{F_n(x) - F(x)\} dx + V_n(t) \quad (2.2)$$

is small for $t \in (0, 1)$ is a slightly more difficult task. Specifically, it is shown in Section 3 that, under the assumptions of the Main Theorem, the statement

$$\sqrt{n/\log \log n} \|V_n\| = o(1), \quad n \rightarrow \infty, \quad (2.3)$$

holds true almost surely. In this section we take (2.3) for granted. We note in passing that, as indicated above, $V_n(1) = 0$. In general, however, for $t \in (0, 1)$, the quantity $V_n(t)$ is not equal to 0. This renders the result (2.3) non-trivial.

Assuming then for the time being (2.3), we now proceed with the proof of the Main Theorem. An elementary calculation on the right-hand side of (2.1) yields the representation

$$\begin{aligned} L_n(t) - L_F(t) &= -\frac{1}{\mu} \int_0^{F^{-1}(t)} \{F_n(x) - F(x)\} dx + \frac{L(t)}{\mu} \\ &\quad \times \int_0^\infty \{F_n(x) - F(x)\} dx + Q_n(t) \end{aligned} \quad (2.4)$$

where $Q_n(t)$ denotes the remainder term:

$$\begin{aligned} &\frac{\mu_n - \mu}{\mu_n \mu} \int_0^{F^{-1}(t)} \{F_n(x) - F(x)\} dx - \frac{\mu_n - \mu}{\mu_n \mu} L(t) \\ &\quad \times \int_0^\infty \{F_n(x) - F(x)\} dx + \frac{1}{\mu_n} V_n(t). \end{aligned}$$

Now, by the classical law of the iterated logarithm, we have that

$$\sqrt{n} \{\mu_n - \mu\} / \sqrt{2 \log \log n} \rightsquigarrow [-\sigma, \sigma] \quad \text{a.s. wrt } |\cdot| \quad \text{on } \mathbf{R}, \quad (2.5)$$

where $\sigma^2 := \text{Var } X$. Furthermore, using assumption (A2), together with Corollary 2 on page 771 of James (1975), we get that the integral $\int |f_n(x) - F(x)| dx$ converges almost surely to 0 as $n \rightarrow \infty$. Therefore, the statement that

$$\sqrt{n/\log \log n} \|Q_n\| = o(1), \quad n \rightarrow \infty, \quad (2.7)$$

holds true almost surely is equivalent to the claim (2.3) (to be proved in Section 3). On the other hand, statement (2.7) in combination with representation (2.4) implies that the Main Theorem amounts to Strassen's (LIL) for the process

$$\begin{aligned} t \mapsto & -\frac{1}{\mu} \int_0^{F^{-1}(t)} \{F_n(x) - F(x)\} dx + \frac{L(t)}{\mu} \\ & \times \int_0^\infty \{F_n(x) - F(x)\} dx, \quad 0 \leq t \leq 1, \end{aligned} \quad (2.8)$$

which we now proceed to prove. To this end we need some additional notation. Let $D[0, \infty)$ denote the set of all bounded and right-continuous functions on $[0, \infty)$ that have left-hand limits at each point. Furthermore, fix a (small) $\delta > 0$, let q_δ be the function $t \mapsto \{t(1-t)\}^{(1/2)-\delta}$, and let ψ_δ be the mapping from $(D[0, \infty), \|\cdot\|)$ into $(D[0, 1], \|\cdot\|)$ defined by

$$\psi_\delta(v)(t) := \frac{1}{\mu} \int_0^{F^{-1}(t)} v(x) q_\delta \circ F(x) dx - \frac{L(t)}{\mu} \int_0^\infty v(x) q_\delta \circ F(x) dx$$

for functions $v \in D[0, \infty)$. If we take $\delta > 0$ sufficiently small (depending on $\varepsilon > 0$ that appears in assumption (A2)), then we have that the integral $\int q_\delta \circ F(x) dx$ is finite. Thus, for small $\delta > 0$,

$$\text{the mapping } \psi_\delta \text{ is continuous.} \quad (2.9)$$

Since the process

$$W_n := \sqrt{\frac{n}{2 \log \log n}} \frac{F_n - F}{q_\delta \circ F}$$

is an element of $D[0, \infty)$, we have by Theorem on page 770 of James (1975) (cf. also Theorem 1 on page 517 of Shorack and Wellner, 1986) that

$$W_n \rightsquigarrow \mathcal{H}_\delta \quad \text{a.s. wrt } \|\cdot\| \quad \text{on } D[0, \infty), \quad (2.10)$$

where

$$\mathcal{H}_\delta := \{h \circ F / q_\delta \circ F : h \in \mathcal{H}\}.$$

Furthermore, by (2.9), (2.10), and the “ \rightsquigarrow mapping theorem” (cf. Theorem 5 on page 78 of Shorack and Wellner, 1986), we get

$$\psi_\delta(W_n) \rightsquigarrow \psi_\delta(\mathcal{H}_\delta) \quad \text{a.s. wrt } \|\cdot\| \quad \text{on } D[0, 1]. \quad (2.11)$$

The simple observation

$$\psi_\delta(\mathcal{H}_\delta) = \{l_h : h \in \mathcal{H}\} \equiv \mathcal{L}$$

completes the proof of Strassen's LIL for the process (2.8). This, in turn, yields the Main Theorem as well. ■

3. THE VERVAAT PROCESS AND POLONIK'S PROOF OF (2.3)

The “expansion” in (2.2) actually defines the process

$$V_n(t) := \int_0^t \{F_n^{-1}(s) - F^{-1}(s)\} ds + \int_0^{F^{-1}(t)} \{F_n(x) - F(x)\} dx, \quad 0 \leq t \leq 1,$$

whose $(0, 1)$ -uniform version is known in the literature as the integrated empirical difference process or, briefly the Vervaat processes (cf., for example, Shorack and Wellner, 1986). The process V_n (in the $(0, 1)$ -uniform case) was introduced and investigated by Vervaat (1972) (cf. also Section 2 in Chapter 15 of Shorack and Wellner, 1986). In particular, Vervaat (1972) proved Strassen's law of the iterated logarithm for the process V_n in the $(0, 1)$ -uniform case, which easily implies that the statement

$$\sqrt{n/\log \log n} \|V_n\| = O(\sqrt{n^{-1} \log \log n}), \quad n \rightarrow \infty, \quad (3.1)$$

holds true almost surely (compare (3.1) with (2.3)).

In Section 2 we faced the crucial (for this paper) problem of showing that statement (2.3) holds true under the conditions of the Main Theorem. The following very beautiful and elegant proof of this fact is due to Wolfgang Polonik.

Proof of (2.3) (Due to W. Polonik). It follows from elementary geometrical considerations (see, for example, Fig. 1 on p. 585 and Eq. (a) on page 594 of Shorack and Wellner, 1986) that

$$0 \leq V_n(t) = \int_{F_n^{-1}(t)}^{F^{-1}(t)} \{F_n(x) - t\} dx \leq |F_n \circ F^{-1}(t) - t| |F_n^{-1}(t) - F^{-1}(t)|. \quad (3.2)$$

Dividing and multiplying the right hand side of (3.2) by the weight function $q_\delta(t) := \{t(1-t)\}^{(1/2)-\delta}$ with some (small) $\delta > 0$, and then taking the

supremum, we get from (3.2) that statement (2.3) is an elementary consequence of the following two facts: (1) Corollary 2 on page 771 of James (1975); and (2) Theorem 3 on page 510 of Mason (1982) (cf. also Exercise 5 on page 651 of Shorack and Wellner, 1986) which implies, in particular, that

$$\|q_\delta\{F_n^{-1} - F^{-1}\}\| = o(1), \quad n \rightarrow \infty,$$

holds true almost surely for some small $\delta > 0$ depending on $\varepsilon > 0$. ■

Remark 3.1. A careful inspection of the bound (3.2) shows that the right-hand side of it reflects the *true* asymptotic behavior of the Vervaat process $V_n(t)$. This fact therefore suggests that the a.s. $o(1)$ rate of convergence in (2.3) can hardly be, in general, increased without postulating more smoothness than the mere continuity of the functions F and F^{-1} . This, in turn, suggests the following quite intriguing

OPEN PROBLEM. Under what conditions on the distribution function F does the result

$$\sqrt{n/\log \log n} \|V_n\| = o(c_n) \quad \text{or} \quad O(c_n), \quad n \rightarrow \infty,$$

hold true almost surely (or in probability) for a fixed sequence c_n , $n = 1, 2, \dots$?

Let us note in concluding that Vervaat (1982) and result (2.3) are special solutions of the Open Problem.

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